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A Stochastic Model of "Market Organization and Trading Relationships"

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1 Introduction

Many markets are characterized by trading relationships. In some markets, buyers trade with particular sellers, and in others, buyers search for sellers for every transaction. In this regard, Weisbuch, Kirman, and Herreiner ([1], henceforth WKH) examine how a market is organized in terms of trading relationships. The authors focus on buyers, experiences gained from past transactions, and construct an evolutionary model of buyers' behavior in a perishable goods market.

The WKH study focuses on a perishable good because buyers' experiences may play an important role in forming trading relationships. We can explain the importance of buyers' experiences in two ways. First, because the goods are perishable, buyers regularly enter the a market and transactions take place frequently. This implies that a good experience will result in buyers choosing the same sellers, whereas a bad experience will cause them to search for other sellers. Second, sellers cannot store perishable goods from one period to the next. Therefore, they only supply the quantities they expect to be able to sell in one period. Buyers face a risk of not being served at all, which is represented by $1 - \Pr(q_i > 0)$ in the WKH model. In such a situation, the experience of past transactions is important when choosing sellers.

The main result of the WKH model is as follows: assuming that buyers learn from their own previous experiences, there is a sharp division between buyers who learn to be loyal to certain sellers (ordered regime) and buyers who continue to “shop around” (disordered regime). This result is supported by simulations and empirical data taken from the Marseille fish market.

Here, we reformulate the WKH model in an evolutionary game theoretic framework. The WKH model loses its attractiveness by employing the mean field approach. Even though the model has stochastic components, the mean field approach replaces them with deterministic components. As Young [2] and Young and Burke [3] show, stochastic variables or errors play an important role in evolutionary models. An excellent feature of their studies is that they solve stochastic models without recasting them as a deterministic model. In this sense, their analysis is successful in utilizing stochastic components.

By following Young [2] and Young and Burke [3], we construct a local interaction model between buyers and sellers. While sellers do not have strategies in the WKH model, the proposed model allows sellers to have two strategies and to construct a 2×2 symmetric game. The probability of no shortage, $\Pr(q_i > 0)$, is reflected in the payoffs of this game. Employing these modifications, we solve the WKH model in a stochastic way.

2 The WKH model

There are n buyers and m sellers in a perishable market. Because a seller, j, cannot store perishable goods from one period to the next, in each period, he supplies a quantity $Q_j(t)$, equal to the amount sold in the previous period.[1] A buyer, i, visits one seller every day. As long as the seller can supply the good, the buyer purchases quantity $q_i(t)$, which generates a profit of $\pi_i(t)$. In this model, the profit $\pi_i(t)$ is taken as an exogenous variable because the prices are not posted. The same market scenario is repeated each day.

A buyer has to choose one seller each day, and the selection is based on the buyer's past experience. A buyer's decision rule is assumed to be composed of two processes: a learning process and a probabilistic choice process; that is, $P(t) : \mathbf{I}(t) \rightarrow \mathbf{J}(t) \rightarrow \mathbf{\Delta}_m$.[2] A learning process, $\mathbf{I}(t) \rightarrow \mathbf{J}(t)$, is a mapping from an information set that contains all past transactions with different sellers, and the profits associated with those transactions, to a vector of preference coefficients. A probabilistic process, $\mathbf{J}(t) \rightarrow \mathbf{\Delta}_m$, is a mapping from a vector of preference coefficients to the unit simplex.

A buyer forms his preference for sellers based only on his own past transactions. Specifically, a preference coefficient of buyer i to seller j at time t is defined as

$$\begin{aligned} J_{ij}(t) &= (1-\gamma) J_{ij}(t-1) + \pi_{ij}(t) \\ &= \sum_{s=0}^{t} (1-\gamma)^s \pi_{ij}(t-s), \end{aligned} \tag{1}$$

where γ is a discount rate. The first term is the discounted sum of past profits, which implies that a buyer gradually forgets

[1] In this setting, $Q_j(t)$ cannot increase, and only decreases.

[2] This formulation allows us to view this decision rule as a mixed strategy.

past events.[3] The second term shows the process of updating information. If in the current period, a buyer i completes a transaction with a seller j, new information from that transaction, $\pi_{ij}(t)$, is added to the preference coefficient J_{ij}. If there is no transaction between a buyer i and a seller j, $\pi_{ij}(t)$ is zero.

Given a vector of preference coefficients $\mathbf{J}(t)$, a buyer calculates the probabilities of visiting each seller. We assume that a buyer maximizes the weighted sum of the expected discounted sum of profits and the Shannon entropy:

$$\max F_i = \beta \sum_{j=1}^{m} P_{ij} J_{ij} - \sum_{j=1}^{m} P_{ij} \log P_{ij}, \text{ s.t. } \sum_{j=1}^{m} P_{ij} = 1. \tag{2}$$

The Shannon entropy represents the degree to which a buyer favors a search. This objective function shows that the more a buyer favors a search, the smaller the weight β is. A probabilistic choice rule is derived from the first order condition, as follows;

$$P_{ij} = \frac{\exp(\beta J_{ij})}{\sum_{j'} \exp(\beta J_{ij'})} \ \forall i, \ \forall j. \tag{3}$$

When β is equal to zero, $P_{ij} = 1/m$ for every j. This implies that a buyer visits a seller at random. When β goes to infinity, a buyer chooses a seller as a best response, according to his preference coefficient.

Mean Field Approach

In order to analyze the model, WKH use the mean field approach. The idea behind the mean field approach is that we

[3] The presence of the discount factor is derived from psychological observations of human behavior.

replace randomly fluctuating quantities by their average values to avoid fluctuations. Here, we apply this approach to the equation for the preference coefficient (1):

$$\begin{aligned} J_{ij}(t) &= (1-\gamma) J_{ij}(t-1) + \pi_{ij}(t) \\ &\Rightarrow \frac{J_{ij}(t+\tau) - J_{ij}(t)}{\tau} = -\gamma J_{ij}(t) + \pi_{ij}(t) \\ &\Rightarrow \frac{dJ_{ij}}{dt} = -\gamma J_{ij} + E[\pi_{ij}] \text{ as } \tau \to 0. \end{aligned}$$

The mean field approach replaces $\pi_{ij}(t)$ in the second line with the expected profit, $E[\pi_{ij}]$, in the third line.

Because we consider a perishable market, a seller may not have sufficient stock, even if a buyer visits the seller with a positive probability. In this case, the buyer cannot obtain a profit. Thus, $\pi_{ij}(t)$ is a stochastic variable. WKH use the mean field approach in order to transform the stochastic difference equation into a deterministic difference equation by replacing the stochastic variable with its average. Then,

$$\begin{aligned} \frac{dJ_{ij}}{dt} &= -\gamma J_{ij} + \Pr(q_i > 0) \cdot P_{ij} \cdot \pi_{ij} \\ &= -\gamma J_{ij} + \Pr(q_i > 0) \cdot \frac{\exp(\beta J_{ij})}{\sum_{j'} \exp(\beta J_{ij'})} \cdot \pi_{ij}, \qquad (4) \end{aligned}$$

where $\Pr(q_i > 0)$ is the probability that a seller has stock when visited by a buyer. We analyze a stationary state; that is, the buyer's preference coefficients do not change; $dJ_{ij}/dt = 0$.

3 A stochastic modeling of WKH

The WKH model loses its attractiveness by employing the mean field approach. Even though the model has stochastic components, the mean field approach replaces these with deterministic components. As Young [2] and Young and Burke [3] show, stochastic variables or errors play an important role in evolutionary models. An excellent feature of their studies is that they solve stochastic models without recasting them as deterministic models. In this sense, their analysis is successful in utilizing stochastic components.

By following Young [2] and Young and Burke [3], we construct a local interaction model between buyers and sellers.[4] Though in the WKH model, sellers do not have strategies, the proposed model allows sellers to have two strategies and to construct a 2×2 symmetric game. The probability of no shortage, $\Pr(q_i > 0)$, is reflected in the payoffs of this game. Using these modifications, we solve the WKH model in a stochastic way.

In order to show that we can apply the local interaction model to the WKH model, we review the games played on the graphs discussed in Young [2]. Imagine that each agent is situated at a vertex of a graph Γ. The set of m vertices is denoted as V, and the set of undirected edges by E. Each undirected edge $\{i, j\}$ has a positive weight $\omega_{ij} = \omega_{ji}$ that measures its relative importance. Vertices i and j are neighbors if $\{i, j\} \in E$. The set of all neighbors of a given vertex i is denoted by N_i. The state of the process is a vector $\mathbf{x} \in X_0^m$, such that $x_i \in X_0$ is

[4] A log-linear response function (3) is often used in the evolutionary game theory literature. For example, Young [2] and Young and Burke [3] utilize the function to analyze local interactions, such as that in a spatial model.

player i's current action for each $i \in V$. Denote the set of states by Ξ. Assume that the one-period payoff to i in state $\mathbf{x}$ is the weighted sum of the payoffs from playing each of his neighbors: $v_i(\mathbf{x}) = \sum_{j \in N_i} \omega_{ij} u(x_i, x_j)$.

The adaptive learning process is defined as the log-linear response rule: $p_i^\beta\left(z \left| \mathbf{x}^t\right.\right) \propto e^{\beta v_i\left(z, \mathbf{x}_{-i}^t\right)}$, where β is a coefficient. Let $P^{\Gamma,\beta}$ be the transition matrix of the associated Markov process. Because $P^{\Gamma,\beta}$ is irreducible, it has a unique invariant distribution, $\mu^{\Gamma,\beta}(\mathbf{x})$. Assume this game has a potential function $\rho(\cdot)$. Then we obtain the following theorem and corollary.

Theorem 1 (Young [2]) *Let G be a symmetric potential game with potential function ρ, and let Γ be a finite, weighted graph. For every $\beta > 0$, the spatial adaptive process $P^{\Gamma,\beta}$ has the unique stationary distribution*

$$\mu^{\Gamma,\beta}(\mathbf{x}) = e^{\beta\rho^*(\mathbf{x})} / \sum_{z \in \Xi} e^{\beta\rho^*(z)},$$

and the stochastically stable states of the spatial game are those that maximize $\rho^*(\mathbf{x})$.

Corollary 1 (Young [2]) *Let G be a symmetric 2×2 coordination game and let Γ be a weighted finite graph. In the spatial game, the stochastically stable states of the adaptive process $P^{\Gamma,\beta}$ are those in which every connected component is coordinated on a risk-dominant equilibrium.*

Now, following the WKH model, we construct a local interaction model. There are n buyers and m sellers in a perishable market. A seller has two strategies in terms of managing his stock: supply a fixed amount or supply the average amount sold in the past. A buyer also has two strategies: being loyal to certain sellers or searching for new sellers. For convenience, we refer to these strategies as follows: L (being loyal), S (search), F (supply a fixed amount), and A (supply the average amount). The history of strategies both agents have employed in the past correspond to the state $\mathbf{x}$.

The weighted sum of the payoffs from playing each of his neighbors, $v_i(\mathbf{x}) = \sum_{j \in N_i} \omega_{ij} u(x_i, x_j)$, can be replaced by the preference coefficient, as follows:

$$J_{ij}(t) = (1-\gamma) J_{ij}(t-1) + \pi_{ij}(t) = \sum_{s=0}^{t} (1-\gamma)^s \pi_{ij}(t-s).$$

The weight ω may be considered the discount factor $1-\gamma$, and the utility function $u(x_i, x_j)$ is an expected utility that is the product of the payoff and the probability of no shortage, $\Pr(q_i > 0)$. The adaptive learning process is defined as the log-linear response rule

$$P_{ij} = \frac{\exp(\beta J_{ij})}{\sum_{j'} \exp\left(\beta J_{ij'}\right)},$$

which is derived from the maximization problem shown in (2). The coefficient β represents the importance buyers place on searching. Then, we can recast the WKH model as the local interaction model mentioned above.

To make the computation of the limit distribution easy, we specify the probability of no shortage as follows:

	F (fixed amount)	A (a*average* of the past)
L (be loyal)	$\Pr(q_i > 0) = 1$	$\Pr(q_i > 0) = c$
S (search)	$\Pr(q_i > 0) = d$	$\Pr(q_i > 0) = b$

Here, $b, c, d \in [0, 1)$. First we assume that buyer is loyal to sellers. Then, if sellers know who their customers are and supply a fixed amount of goods, the buyers always buy when they visit the sellers to whom they are loyal. If sellers supply the average amount, even though they know their customers, there is some risk that a buyer will not be able to buy, so the probability is set as $\Pr(q_i > 0) = c$. Next we assume that a buyer searches for sellers. Then, there is some risk of a shortage of goods, regardless of the sellers' strategies.

For simplicity, consider a homogeneous products case, that is, $\pi_j = \pi$, for all $j = \{1, 2, ..., m\}$. Each time, one buyer and one seller are chosen randomly. Their payoffs can be represented as expected profits as follow:

	F	A
L	$\pi,\ \pi$	$c\pi,\ d\pi$
S	$d\pi,\ c\pi$	$b\pi,\ b\pi$

The first payoff in each cell is that of the buyer, and the second is that of the seller. Then, this game is a potential game, and ρ can be chosen as follows:

$$\rho(L, F) = (1 - d)\pi, \quad \rho(L, A) = 0,$$
$$\rho(S, F) = 0, \qquad \rho(S, A) = (b - c)\pi.$$

Now, suppose this is a coordination game, with coordination equilibria (L, F) and (S, A); that is, $1 > d$ and $b > c$. Strategy (L, F) is strictly risk dominant if and only if $1 - d > b - c$, and strategy (S, A) is strictly risk dominant if the reverse inequality holds. Thus, it follows that, as $\beta \to \infty$, the stationary distribution places all support in those states in which everyone is coordinated in the risk-dominant equilibrium. By applying corollary 6.1 of Young [2], we obtain the following proposition.

Proposition 1 *Recast the WKH model as a local interaction model. Then, the stochastically stable states of the adaptive process* $P^{\Gamma,\beta}$ *are those in which every connected component is coordinated in a risk-dominant equilibrium.*

Thus, by recasting the original model as a stochastic model, we may obtain richer results. That is, we can have one of the coordination equilibria (L, F) and (S, A) as the stochastically stable state. As we have seen, the original model uses the mean field approach to avoid incorporating stochastic factors. However, by constructing a local interaction model, we can avoid using the mean field approach and provide a framework that can utilize stochastic components in organizing a market.

Notes:

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References

[1] Weisbuch, G., Kirman, A., and Herreiner, D. (2000), “Market organization and trading relationships,” *Economics Journal*, vol. 110, pp. 411–436.

[2] Young, H. P. (1998), *Individual strategy and social structure: An evolutionary theory of institution.* Princeton, NJ: Princeton University Press.

[3] Young, H. P. and Burke, M. A. (2001), "Competition and custom in economic contract: A case study of Illinois agriculture," *American Economic Review*, vol. 91, pp. 559–573.